Estimate of blow-up and relaxation time for self-gravitating Brownian particles and bacterial populations

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Abstract

We determine an asymptotic expression of the blow-up time t_{coll} for self-gravitating Brownian particles or bacterial populations (chemotaxis) close to the critical point. We show that $t_{coll} = t_*(\eta - \eta_c)^{-1/2}$ with $t_* = 0.91767702...$, where η represents the inverse temperature (for Brownian particles) or the mass (for bacterial colonies), and η_c is the critical value of η above which the system blows up. This result is in perfect agreement with the numerical solution of the Smoluchowski-Poisson system. We also determine the asymptotic expression of the relaxation time close but above the critical temperature and derive a large time asymptotic expansion for the density profile exactly at the critical point.

1 Introduction

Recently, several papers have focused on the blow-up properties of a cluster of self-attracting particles [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. This concerns in particular the "isothermal collapse" [13] of self-gravitating Brownian particles and the "chemotactic aggregation" [14] of bacterial populations in biology. These systems have a similar mathematical structure and they are described, in a first approximation, by the Smoluchowski-Poisson system [15]. The Smoluchowski equation is a particular Fokker-Planck equation in physical space involving a diffusion due to Brownian motion and a drift. In the standard Brownian theory developed by Einstein, the drift is due to an external potential [16]. Alternatively, we can consider the more complicated situation in which the potential is produced by the particles themselves. If we assume a Newtonian type of interaction, we have to couple the Smoluchowski equation to the Poisson equation.

It can be shown that the Smoluchowski-Poisson system decreases a Lyapunov functional $F[\rho]$ which can be interpreted as a Boltzmann free energy [17]. Furthermore, the type of evolution depends on the value of a dimensionless parameter η. For $\eta \leq \eta_c = 2.51755132...$, the Smoluchowski-Poisson system evolves to an equilibrium state which minimizes the free energy at fixed mass. It corresponds to an isothermal distribution of particles similar to isothermal stars and isothermal stellar systems [18]. On the contrary, for $\eta > \eta_c$, there is no possible

equilibrium state and the system blows up. This is the case for self-gravitating Brownian particles below a critical temperature T_c and for bacterial populations above a critical mass M_c. It is found that the collapse is self-similar and that it develops a finite time singularity, *i.e.* the central density becomes infinite in a finite time t_{coll} [7, 8]. Then, a Dirac peak is formed in the post-collapse regime [11].

It is clear that the collapse time t_{coll} depends on the distance to the critical point η_c and that it should diverge as $\eta \to \eta_c$. More precisely, by using heuristic arguments, it is argued in [7] that $t_{coll} \sim t_*(\eta_c - \eta)^{-1/2}$. This scaling is consistent with numerical simulations of the Smoluchowski-Poisson system. However, the approach of [7] is qualitative and does not provide the numerical value of t_*.

After some introductory material presented in Section 2, we develop a systematic procedure in Section 3, inspired from [19], which confirms the scaling law $t_{coll} \sim t_*(\eta_c-\eta)^{-1/2}$ and leads to the explicit value of t_*. In section 4, we study the relaxation time τ to equilibrium below η_c and show that it diverges like $\tau \sim c_\eta(\eta-\eta_c)^{-1/2}$, where c_η is given explicitly. In Section 5, we derive a systematic large time expansion of the density profile exactly at η_c. In particular, we show that the central density approaches its equilibrium value according to $\rho_0 - \rho(r=0,t) \sim \frac{c_\rho}{t}$, where c_ρ is a universal constant which is explicitly computed. Finally, in Section 6, we determine the pulsation period of bounded isothermal spheres described by the Euler-Jeans equations close to the critical point.

2 Self-gravitating Brownian particles and bacterial populations

2.1 The model equations

In the overdamped regime, a system of self-gravitating Brownian particles is described by the N coupled stochastic equations

$$\frac{d\mathbf{r}_i}{dt} = -\mu\nabla_i U(\mathbf{r}_1,...,\mathbf{r}_N) + \sqrt{2D}\,\mathbf{R}_i(t), \tag{1}$$

where $\mu = 1/\xi$ is the mobility (ξ is the friction coefficient), D is the diffusion coefficient and $\mathbf{R}_i(t)$ is a white noise satisfying $\langle \mathbf{R}_i(t)\rangle = \mathbf{0}$ and $\langle R_{a,i}(t)R_{b,j}(t')\rangle = \delta_{ij}\delta_{ab}\delta(t-t')$, where $a,b = 1,2,3$ refer to the coordinates of space and $i,j = 1,...,N$ to the particles. We define the temperature $T = 1/\beta$ through the Einstein relation $\mu = D\beta$. The particles interact via the potential $U(\mathbf{r}_1,...,\mathbf{r}_N) = \sum_{i<j} u(\mathbf{r}_i-\mathbf{r}_j)$. In this paper, $u(\mathbf{r}_i-\mathbf{r}_j) = -G/|\mathbf{r}_i-\mathbf{r}_j|$ is the Newtonian binary potential in $d=3$ dimensions. Starting from the N-body Fokker-Planck equation and using a mean-field approximation [20, 21], we can derive the nonlocal Smoluchowski equation

$$\frac{\partial\rho}{\partial t} = \nabla\left[\frac{1}{\xi}(T\nabla\rho + \rho\nabla\Phi)\right], \tag{2}$$

where the potential $\Phi(\mathbf{r},t)$ is produced by the density $\rho(\mathbf{r},t)$ according to the Poisson equation

$$\Delta\Phi = 4\pi G\rho. \tag{3}$$

The self-gravitating Brownian gas has a rigorous canonical thermodynamic structure [7]. It can be seen therefore as the canonical counterpart of the usual N-stars problem governed by Newton's equations and possessing a microcanonical structure.

On the other hand, in a biological context, a model of chemotactic aggregation exhibiting blow-up phenomena is provided by the system of equations

$$\frac{\partial \rho}{\partial t} = D\Delta\rho - \chi\nabla(\rho\nabla c), \tag{4}$$

$$\Delta c = -\lambda\rho, \tag{5}$$

where ρ is the concentration of the biological population (amoebae), c is the concentration of the substance secreted (acrasin) and χ measures the strength of the chemotactic drift. Equations (4) and (5) can be viewed as a simplification of the Keller-Segel model [14]. Clearly, the two models are isomorphic provided that we make the identification $\Phi \leftrightarrow -\frac{4\pi G}{\lambda}c$, $T \leftrightarrow \frac{4\pi GD}{\lambda\chi}$ and $\xi \leftrightarrow \frac{4\pi G}{\lambda\chi}$. Introducing the mass $M = \int \rho\, d^3\mathbf{r}$ of the system and the radius R of the domain, we can show that the problem depends on the single dimensionless parameter $\eta = \beta GM/R$ [7]. Therefore, a large value of η corresponds to a small temperature T or a large mass M.

2.2 The Smoluchowski-Poisson system

In the following, we shall use the gravitational terminology but we stress that our results are equally valid for the biological problem due to the above analogy. By rescaling the physical parameters adequately, it is possible to take $M = R = G = \xi = 1$ without restriction. Then, the equations of the problem are

$$\frac{\partial \rho}{\partial t} = \nabla(T\nabla\rho + \rho\nabla\Phi), \tag{6}$$

$$\Delta\Phi = 4\pi\rho, \tag{7}$$

with proper boundary conditions in order to impose a vanishing particle flux on the surface of the confining sphere. If we restrict ourselves to spherically symmetric solutions, the boundary conditions are

$$\frac{\partial \Phi}{\partial r}(0,t) = 0, \qquad \Phi(1,t) = -1, \qquad T\frac{\partial \rho}{\partial r}(1,t) + \rho(1,t) = 0. \tag{8}$$

With this rescaling, the equations only depend on the temperature T, or equivalently on the parameter $\eta = 1/T$. As mentioned previously, a small temperature is equivalent (through a rescaling) to a large mass.

Integrating Eq. (7) once, we can rewrite the Smoluchowski-Poisson system in the form of a single integrodifferential equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{ r^2 \left(T\frac{\partial \rho}{\partial r} + \frac{\rho}{r^2}\int_0^r \rho(r',t) 4\pi r'^2\, dr' \right) \right\}. \tag{9}$$

The Smoluchowski-Poisson system is also equivalent to a single differential equation

$$\frac{\partial M}{\partial t} = T\left(\frac{\partial^2 M}{\partial r^2} - \frac{2}{r}\frac{\partial M}{\partial r}\right) + \frac{1}{r^2}M\frac{\partial M}{\partial r}, \tag{10}$$

for the quantity

$$M(r,t) = \int_0^r \rho(r',t) 4\pi r'^2\, dr', \tag{11}$$

which represents the mass contained within the sphere of radius r. The appropriate boundary conditions are

$$M(0,t) = 0, \qquad M(1,t) = 1. \tag{12}$$

3 Asymptotic estimate of the collapse time

3.1 Systematic expansion

We know that the Smoluchowski-Poisson system blows up for $T < T_c$, where T_c is a critical temperature below which there is no equilibrium state [7]. We shall place ourselves close to the critical point and expand the mass profile as

$$M(r,t) = M_c(r) + \epsilon M_1(r,t) + \epsilon^2 M_2(r,t) + ..., \tag{13}$$

where $M_c(r)$ is the equilibrium profile at T_c and ϵ is a small parameter defined as

$$\epsilon = \left(\frac{T_c - T}{T_c}\right)^{1/2} \ll 1. \tag{14}$$

We also rescale the time according to

$$t = \tau/\epsilon. \tag{15}$$

If we substitute the expansion Eq. (13) in Eq. (10) and equal terms of same order, we get to order 0:

$$T_c\left(\frac{\partial^2 M_c}{\partial r^2} - \frac{2}{r}\frac{\partial M_c}{\partial r}\right) + \frac{M_c}{r^2}\frac{\partial M_c}{\partial r} = 0, \tag{16}$$

to order 1:

$$T_c\left(\frac{\partial^2 M_1}{\partial r^2} - \frac{2}{r}\frac{\partial M_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}(M_1 M_c) = 0, \tag{17}$$

and to order 2:

$$\frac{\partial M_1}{\partial \tau} = T_c\left(\frac{\partial^2 M_2}{\partial r^2} - \frac{2}{r}\frac{\partial M_2}{\partial r} - \frac{\partial^2 M_c}{\partial r^2} + \frac{2}{r}\frac{\partial M_c}{\partial r}\right) + \frac{1}{r^2}\left(M_c\frac{\partial M_2}{\partial r} + M_1\frac{\partial M_1}{\partial r} + M_2\frac{\partial M_c}{\partial r}\right). \tag{18}$$

The boundary conditions are

$$M_{n\geq 0}(0,\tau) = 0, \quad M'_{n\geq 0}(0,\tau) = 0, \tag{19}$$

$$M_c(1,\tau) = 1, \quad M_{n\geq 1}(1,\tau) = 0. \tag{20}$$

3.2 Order 0: the equilibrium state $M_c(\xi)$

At equilibrium, the condition of hydrostatic balance $T\nabla\rho + \rho\nabla\Phi = 0$ combined with the Gauss theorem $d\Phi/dr = M(r)/r^2$ leads to the relation

$$M(r) = -Tr^2\frac{d\ln\rho}{dr}. \tag{21}$$

Since $M'(r) = 4\pi\rho r^2$, we obtain the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{T}{\rho}\frac{d\rho}{dr}\right) = -4\pi\rho. \tag{22}$$

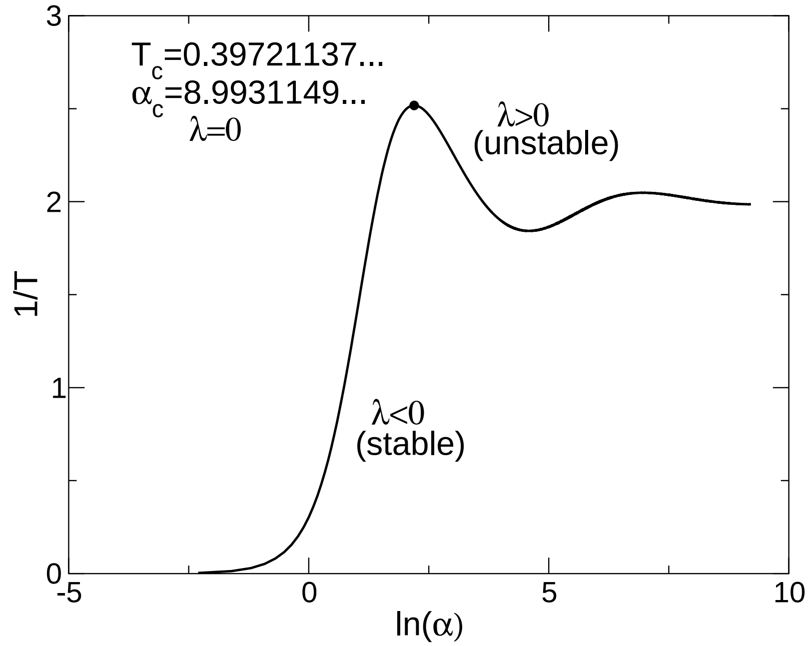

Figure 1: We plot $\eta(\alpha) = T^{-1}(\alpha)$ where $\alpha = (4\pi\rho_0/T)^{1/2}$ parameterizes the series of equilibria. For $\alpha > \alpha_c$, unstable modes with $\lambda > 0$ arise, which are solution of Eq. (33). The minimum temperature T_c corresponds precisely at the point of marginal stability $\lambda = 0$.

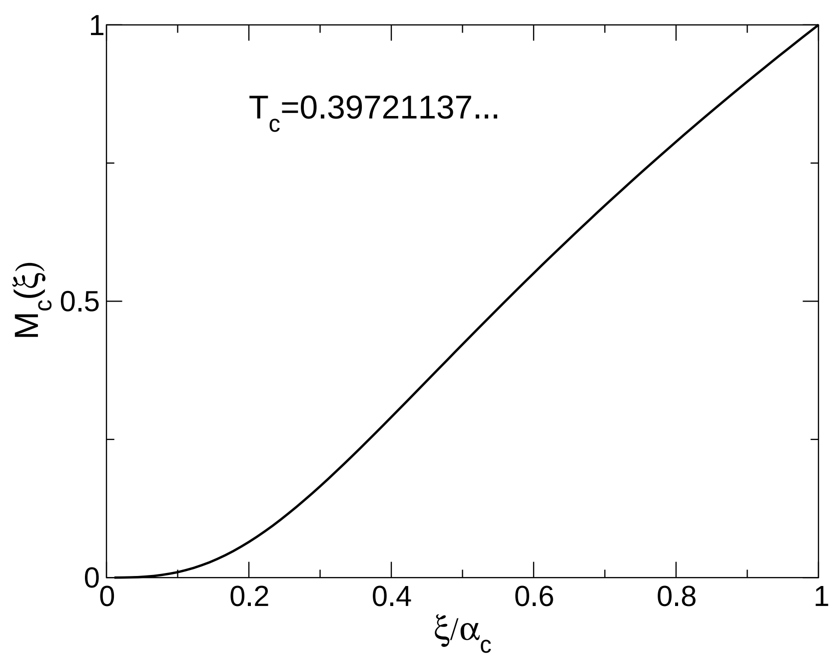

Figure 2: The integrated mass density is plotted at the critical point.

Introducing the function ψ through the relation

$$\rho = \rho_0 e^{-\psi}, \tag{23}$$

where ρ_0 is the central density, and using the normalized distance $\xi = \alpha r$ where $\alpha = (4\pi\rho_0/T)^{1/2}$, we can rewrite the relation Eq. (21) in the form

$$M(\xi) = T\frac{\xi^2}{\alpha}\psi'(\xi). \tag{24}$$

Furthermore, according to Eq. (22), the function ψ is solution of the Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi}, \tag{25}$$

$$\psi(0) = \psi'(0) = 0. \tag{26}$$

The Taylor expansion of the Emden function near the origin is

$$\psi = \frac{1}{6}\xi^2 - \frac{1}{120}\xi^4 + ... \tag{27}$$

Using these results, we can check that the Emden equation (25) is equivalent to Eq. (16), as it should. For a given temperature $T \geq T_c$, the Emden equation has to be solved from $\xi = 0$ to $\xi = \alpha$ such that

$$\alpha\psi'(\alpha) = \frac{1}{T}. \tag{28}$$

For the profile we are considering, at the critical temperature T_c, we have to stop the integration at α_c such that $T(\alpha_c)$ is minimum. The condition $[1/T]'(\alpha_c) = 0$ is equivalent to

$$\frac{\alpha_c\ e^{-\psi(\alpha_c)}}{\psi'(\alpha_c)} = 1. \tag{29}$$

It is found numerically that $\alpha_c = 8.9931149...$ and $T_c = 0.39721137....$

3.3 Order 1: the function $F(\xi)$

Since Eq. (17) is linear, we look for solutions of the form

$$M_1 = A(\tau)F(\xi). \tag{30}$$

The function $F(\xi)$ satisfies the differential equation

$$\frac{d^2F}{d\xi^2} - \frac{2}{\xi}\frac{dF}{d\xi} + \frac{1}{\xi^2}\frac{d}{d\xi}(F\xi^2\psi') = 0, \tag{31}$$

which is equivalent to

$$\frac{d}{d\xi}\left(\frac{e^{\psi}}{\xi^2}\frac{dF}{d\xi}\right) + \frac{F}{\xi^2} = 0, \tag{32}$$

where we have used the Emden equation Eq. (25). Equation (32) is precisely the equation found at the critical point T_c by analyzing the linear stability of isothermal spheres [7]. Indeed, considering a perturbation $\delta M \sim e^{\lambda t}$ around a stationary solution $M(r)$ of Eq. (10), we get

$$T\left(\frac{\partial^2 \delta M}{\partial r^2} - \frac{2}{r}\frac{\partial \delta M}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}(M\delta M) = \lambda \delta M. \tag{33}$$

If $\lambda < 0$, the stationary solution is stable and if $\lambda > 0$, the stationary solution is unstable. At the critical point T_c where $\lambda = 0$ (marginal stability), we recover Eq. (17) with $M_1 = \delta M$. Equation (32) has to be solved with the boundary conditions

$$F(0) = F'(0) = 0. \tag{34}$$

We find that $F(\xi)$ can be simply expressed in terms of $\psi(\xi)$ as [13]:

$$F(\xi) = \xi^3 e^{-\psi} - \xi^2 \psi'. \tag{35}$$

The condition $F(\alpha_c) = 0$ determines the critical temperature T_c. Using Eq. (35), this condition is equivalent to

$$\frac{\alpha_c \; e^{-\psi(\alpha_c)}}{\psi'(\alpha_c)} = 1. \tag{36}$$

Comparing this result with Eq. (29), we find that the criterion of marginal stability ($\lambda = 0$) corresponds to the point α_c in the series of equilibria $T = T(\alpha)$ where the temperature is minimum ($[1/T]'(\alpha_c) = 0$) [13, 7].

3.4 Order 2: the function $A(\tau)$

Using the previous results, Eq. (18) can be rewritten

$$F(\xi)\dot{A}(\tau) = T_c \alpha_c^2 \left[\frac{\partial^2 M_2}{\partial \xi^2} - \frac{2}{\xi}\frac{\partial M_2}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial}{\partial \xi}(M_2 \xi^2 \psi')\right] + \alpha_c T_c^2 \xi^2 \psi' e^{-\psi} + \frac{\alpha_c^3}{\xi^2} A^2 F \frac{dF}{d\xi}. \tag{37}$$

We now multiply Eq. (37) by a function $\chi(\xi)$ and integrate between 0 and α_c. We choose the function χ (see below) so as to eliminate the terms where M_2 appears. We are thus left with

$$\dot{A} = KA^2 + B, \tag{38}$$

where

$$K = \frac{\alpha_c^3 \int_0^{\alpha_c} \frac{\chi}{\xi^2} F \frac{dF}{d\xi} d\,\xi}{\int_0^{\alpha_c} \chi F \, d\xi}, \qquad B = \alpha_c T_c^2 \, \frac{\int_0^{\alpha_c} \chi \xi^2 \psi' e^{-\psi} d\,\xi}{\int_0^{\alpha_c} \chi F \, d\xi}. \tag{39}$$

For future convenience, we note that

$$F'(\xi) = \xi^2 e^{-\psi}(2 - \xi \psi'). \tag{40}$$

By construction, the function χ satisfies the integral relation

$$\int_0^{\alpha_c} \left[\frac{\partial^2 M_2}{\partial \xi^2} - \frac{2}{\xi}\frac{\partial M_2}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial}{\partial \xi}(M_2 \xi^2 \psi')\right] \chi(\xi) \, d\xi = 0. \tag{41}$$

Integrating by parts and using $M_2(0) = M_2'(0) = M_2(\alpha) = 0$, we obtain

$$\chi(\alpha_c)M_2'(\alpha_c, \tau) + \int_0^{\alpha_c} \left\{ \frac{d^2\chi}{\partial\xi^2} + \left(\frac{2}{\xi} - \psi'\right)\frac{d\chi}{d\xi} + \frac{2}{\xi^2}(\xi\psi' - 1)\chi \right\} M_2(\xi, \tau)\, d\xi = 0. \tag{42}$$

Thus, we impose that χ is a solution of the differential equation

$$\frac{d^2\chi}{\partial\xi^2} + \left(\frac{2}{\xi} - \psi'\right)\frac{d\chi}{d\xi} + \frac{2}{\xi^2}(\xi\psi' - 1)\chi = 0, \tag{43}$$

with the boundary condition

$$\chi(0) = 0. \tag{44}$$

It turns out that the solution of this equation can be expressed in terms of the Emden function as

$$\chi(\xi) = \frac{1}{\xi^2}F(\xi)e^{\psi} = \xi - \psi' e^{\psi}, \tag{45}$$

with $\chi'(0) = 2/3$. The function $\chi(\xi)$ is represented in Fig. 3 along with the function $F(\xi)$. Using Eq. (36), we readily check that $\chi(\alpha_c) = 0$. Therefore, the condition Eq. (41) is satisfied. Accordingly, $A(\tau)$ is determined by Eq. (38), where K and B have to be evaluated using Eq. (39).

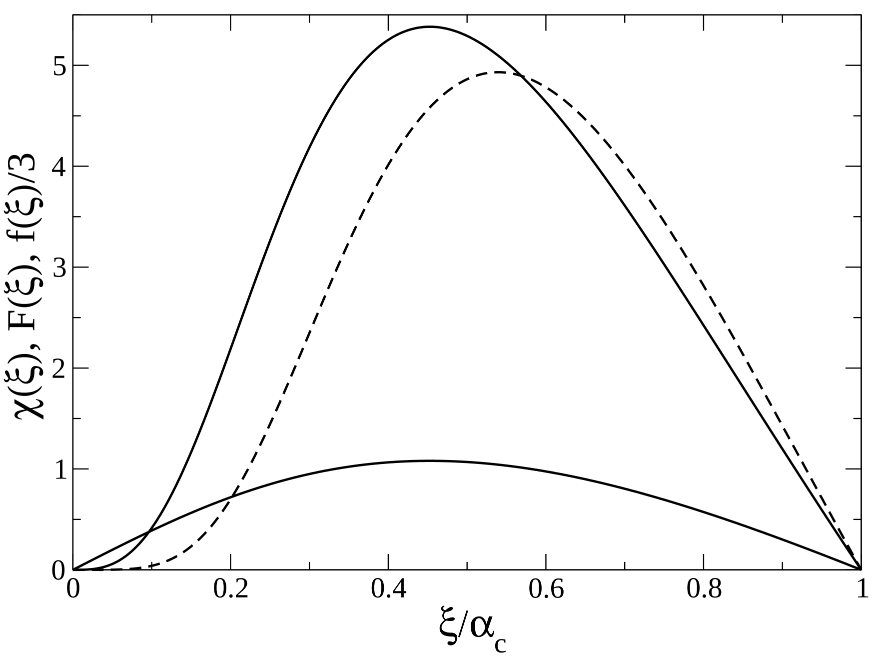

Figure 3: We plot $\chi(\xi)$ (lower curve), $F(\xi)$ (upper full line) and the eigensolution $f(\xi)$ of Eq. (56) (dashed line ; $f(\xi)$ has been divided by a factor 3 for convenience).

3.5 The blow up time

The solution of Eq. (38) with $A(\tau) \to -\infty$ as $\tau \to 0$ is

$$A(\tau) = \left(\frac{B}{K}\right)^{1/2} \tan\left[\tau (BK)^{1/2} - \frac{\pi}{2}\right]. \tag{46}$$

Returning to the original time variable, we get

$$A(t) = \left(\frac{B}{K}\right)^{1/2} \tan\left[t(1 - T/T_c)^{1/2} (BK)^{1/2} - \frac{\pi}{2}\right]. \tag{47}$$

Then, we find that the blow up occurs for

$$t_{coll} = t_*(\eta - \eta_c)^{-1/2}, \tag{48}$$

with

$$t_* = \pi \left(\frac{\eta_c}{BK}\right)^{1/2}. \tag{49}$$

We find numerically that $K = 62.56038...$ and $B = 0.4716274....$ We thus get

$$t_{coll} = t_*(\eta - \eta_c)^{-1/2}, \qquad t_* = 0.91767702... \tag{50}$$

This asymptotic result is compared in Fig. 4 with the exact law $t_{coll}(T)$ obtained by solving numerically the Smoluchowski-Poisson system.

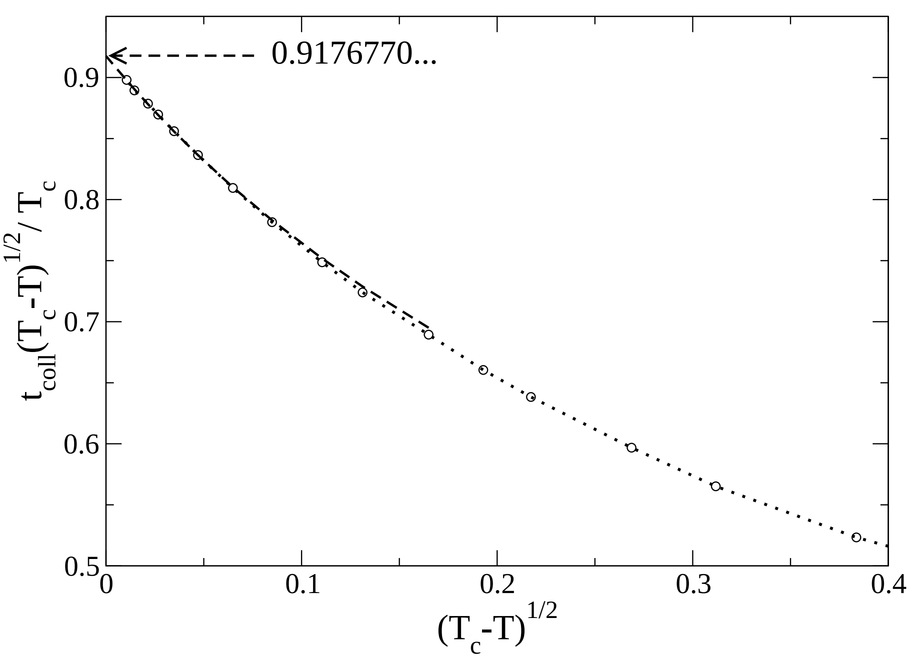

Figure 4: We plot $t_{coll} \times (T_c - T)^{1/2}/T_c$ computed numerically as a function of $(T_c - T)^{1/2}$, which should converge to $t_* = 0.91797702....$ A quadratic fit in the region $(T_c - T)^{1/2} < 0.05$ retrieves the first four digits of t_*.

4 Relaxation time estimate for $T > T_c$

We now assume $T > T_c$, so that an equilibrium state exists. We look for a time dependent solution for $M(r,t)$ which converges exponentially to the equilibrium profile

$$M(r,t) = M_T(r) + F_T(r)\mathrm{e}^{-t/\tau}. \tag{51}$$

The purpose of this section is to compute the leading asymptotic form of the relaxation time $\tau(T) \to +\infty$ as T goes to T_c. After inserting this *ansatz* into the equation of motion for $M(r,t)$, we obtain

$$F_T'' + \left(\phi' - \frac{2}{r}\right) F_T' + \alpha^2 \mathrm{e}^{-\phi} F_T = -(\tau T)^{-1} F_T, \tag{52}$$

where

$$\phi(r) = \psi(\alpha r). \tag{53}$$

We have used r as a coordinate (instead of ξ) so as to remain within a fixed interval of space $0 \le r \le 1$ as we make the perturbative expansion in α (see below). When $T \to T_c$, we have $\alpha \to \alpha_c$, $\tau \to +\infty$, and $F_T(r) \to F(\alpha_c r)$, where $F(\xi)$ is given by Eq. (35). We now expand Eq. (52) in power of $\varepsilon = \frac{\alpha_c - \alpha}{\alpha_c}$, by introducing $f(\xi)$ and μ such that

$$F_T(r) = F(\alpha_c r) + \varepsilon f(\alpha_c r) + O(\varepsilon^2). \tag{54}$$

and

$$(\tau \alpha_c^2 T_c)^{-1} = \mu\varepsilon + O(\varepsilon^2). \tag{55}$$

The relation between $T - T_c$ and ε will be given later. Collecting terms proportional to ε, and setting $\xi = \alpha_c x$, we obtain

$$f'' + \left(\psi' - \frac{2}{\xi}\right) f' + f\mathrm{e}^{-\psi} - \frac{2}{\xi^2} F F' = -\mu F, \tag{56}$$

subject to the boundary condition $f(0) = f(\alpha_c) = 0$. This eigenvalue problem selects a unique value for μ, hence for τ. We were not able to solve this problem analytically. However, we remark that this equation is similar to Eq. (37), without the term leading to B. We can therefore use the same trick as in Sec. 3.4. After multiplying Eq. (56) by the function χ defined in the previous section, we end up with

$$\mu = \frac{\int_0^{\alpha_c} \frac{2\chi(\xi)}{\xi^2} F(\xi) F'(\xi)\, d\xi}{\int_0^{\alpha_c} \chi(\xi) F(\xi)\, d\xi} = \frac{2K}{\alpha_c^3} = 0.17202792... \tag{57}$$

The direct numerical solution of Eq. (56) leads of course to the same value. The function $f(\xi)$ is plotted in Fig. 3.

We now find a simple relation between $T - T_c$ and ε close to T_c [13]. Using the condition $\alpha\psi'(\alpha) = 1/T$ and the fact that $(\xi\psi')'_{|\alpha_c} = 0$, we obtain

$$\frac{1}{T_c} - \frac{1}{T} = \frac{(\alpha_c - \alpha)^2}{2} (\xi\psi')''_{|\alpha_c} + O((\alpha_c - \alpha)^3), \tag{58}$$

which up to leading order in ε can be written in the form

$$\varepsilon \sim \sqrt{\frac{2(T-T_c)}{T_c(1-2T_c)}}. \tag{59}$$

Finally, we obtain the following diverging behavior for τ:

$$\tau^{-1} \sim \mu\sqrt{\frac{2T_c(T-T_c)}{1-2T_c}}\alpha_c^2. \tag{60}$$

Numerically, we get

$$\tau \sim c_T(T-T_c)^{-1/2} \sim c_\eta(\eta_c-\eta)^{-1/2} \tag{61}$$

with

$$c_T = 0.036563056..., \qquad c_\eta = 0.09204937... \tag{62}$$

Note that a heuristic argument leading to $\tau \sim (T-T_c)^{-1/2}$ was given in [7], although the constant c_T could not be calculated.

5 The decay of the density profile at $T = T_c$

Strictly at $T = T_c$, $\tau = +\infty$, and one expects a slow convergence to the equilibrium profile $M_c(r)$. Let us write

$$M(r,t) = M_c(r) - A(t)M_1(r) - B(t)M_2(r) + ... \tag{63}$$

We implicitly assume that $B(t) \ll A(t)$ as $t \to +\infty$. Moreover, the presence of the minus signs in Eq. (63) anticipates the fact that starting from a flat density the central density increases. In [7], it was argued based on a heuristic argument that $A(t) \sim t^{-1}$. It is the purpose of this section to justify this statement as well as to give more quantitative estimates.

Let us postulate for the moment the natural choice $B(t) \sim A(t)^2$ which will be justified hereafter. Inserting again this *ansatz* in the dynamical equation for $M(r,t)$, and collecting terms of order $A(t)$ and $A(t)^2$, we immediately find that the first equation leads to

$$M_1(r) = \sqrt{\frac{2T_c}{\alpha_c}}F(\alpha_c r), \tag{64}$$

where the constant has been set for later convenience, as multiplying M_1 and dividing A by the same constant leaves the expansion for M unchanged. Setting $M_2(r) = g(\alpha_c r)$, we find that g satisfies

$$g'' + \left(\psi' - \frac{2}{\xi}\right)g' + g\mathrm{e}^{-\psi} - \frac{2}{\xi^2}FF' = -\lambda F, \tag{65}$$

with

$$\lambda = -\frac{\dot{A}}{A^2}\sqrt{\frac{2\alpha_c}{T_c}}\alpha_c^{-3}, \tag{66}$$

which must be a constant independent of time. Thanks to the proper choice of the constant in Eq. (64), we find that Eq. (65) and Eq. (56) are identical. The only physical solution with $g(0) = g(\alpha_c) = 0$ corresponds to the eigenvalue $\lambda = \mu$. Finally, we find

$$A(t) \sim t^{-1}\mu^{-1}\sqrt{\frac{2\alpha_c}{T_c}}\alpha_c^{-3}. \tag{67}$$

Once the signs have been fixed in Eq. (63), we indeed find that the small time correction to the central density is necessarily negative, as M_1, M_2 (or g) and A are found to be positive. Therefore, the equilibrium profile $M_c(r)$ is approached from below, which is natural since an excess of mass would provoke gravitational collapse.

Let us illustrate this result by calculating the correction to the central density. Using Eq. (64) and Eq. (67), and the fact that $M_c(r) \sim \frac{4\pi}{3}\rho_0 r^3$ and $F(\xi) \sim \frac{2}{3}\xi^3$, we find that for large time the central density converges to ρ_0 from below in a universal manner

$$\rho_0 - \rho(r = 0, t) = \frac{c_\rho}{t} + O(t^{-2}), \tag{68}$$

with

$$c_\rho = (\pi\mu)^{-1} = 1.8503385... \tag{69}$$

This result is illustrated quantitatively in Fig. 5.

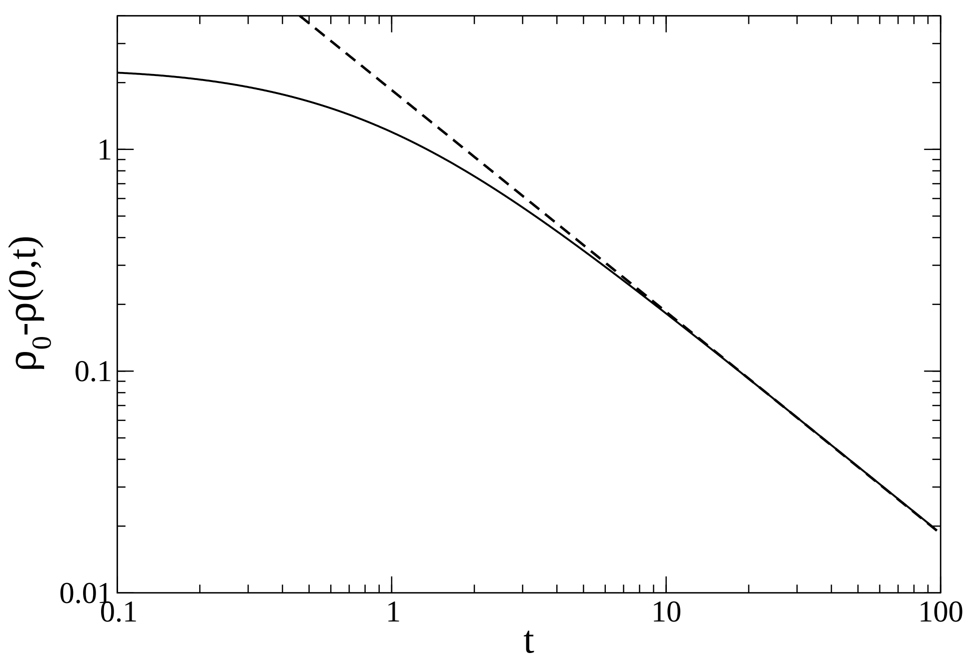

Figure 5: At $T = T_c$, and starting from a uniform mass density, we have computed numerically $\rho_0 - \rho(r = 0, t)$, where ρ_0 is the equilibrium central density. This is compared with the exact and universal large time estimate c_ρ/t, with $c_\rho = (\pi\mu)^{-1} = 1.8503385...$ (dashed line).

Finally, we discuss the *a priori* unjustified choice $B(t) \sim A^2(t)$. The case $B(t) \ll A^2(t)$ is clearly impossible as it amounts to take $g = 0$ in Eq. (65), leading to an equation which is never satisfied by F, whatever the choice for $A(t)$. The opposite case $B(t) \gg A^2(t)$ leads to an

equation similar to Eq. (65) but where the term proportional to FF' is absent. For this equation, it is clear that if a solution g_0 exists for a specific λ_0, then $\lambda/\lambda_0 g_0$ is also solution associated to λ. This implies that the only solution satisfying the boundary condition $g(0) = g(\alpha_c) = 0$ is actually the null function associated to $\lambda = 0$, which is not permitted for $T > T_c$. Hence $B(t) \sim A^2(t)$ is the only possibility leading to a physically sensible solution.

6 The pulsation period of isothermal spheres

In Sec. 4, we have determined the relaxation time of self-gravitating Brownian particles described by the Smoluchowski-Poisson system close to the critical point T_c. Writing the perturbation as $\delta\rho \sim e^{\lambda t}$ where $\lambda = -1/\tau$, the eigenvalue equation for λ can be written in dimensional form as [7]:

$$\frac{d}{dr}\left(\frac{1}{4\pi\rho r^2}\frac{dq}{dr}\right) + \frac{Gq}{Tr^2} = \frac{\lambda\xi}{4\pi\rho T r^2}q, \tag{70}$$

where $q(r) = \delta M(r)$. This equation is the same as Eq. (52). Using the result of Sec. 4, and returning to dimensional parameters, the inverse relaxation time is given by

$$\lambda = \mp\frac{1}{c_\eta}(\eta_c - \eta)^{1/2}\frac{GM}{\xi R^3}. \tag{71}$$

The sign $-$ corresponds to $\alpha < \alpha_c$ and leads to exponentially damped pertubations (stable). The sign $+$ corresponds to $\alpha > \alpha_c$ and leads to exponentially growing pertubations (unstable).

We now consider the case of gaseous self-gravitating systems (stars) described by the Euler-Jeans equations:

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0, \tag{72}$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{73}$$

$$\Delta\Phi = 4\pi G\rho. \tag{74}$$

We use an isothermal equation of state $p = \rho T$ and we assume that the system is confined within a region of size R. These equations provide the usual starting point for the analysis of the gravitational stability of gaseous systems. The case of an infinite homogeneous medium was first investigated by Jeans [22] in his classical study. The case of a finite inhomogeneous medium was considered by Chavanis [13]. In that case, the equation of pulsation can be written in the form

$$\frac{d}{dr}\left(\frac{1}{4\pi\rho r^2}\frac{dq}{dr}\right) + \frac{Gq}{Tr^2} = \frac{\lambda^2}{4\pi\rho T r^2}q, \tag{75}$$

where, as before, $\delta\rho \sim e^{\lambda t}$. Comparing with Eq. (70), we see that the results of Sec. 4 remain valid provided that $\lambda\xi$ is replaced by λ^2. Therefore, Eq. (71) translates into

$$\lambda^2 = \mp\frac{1}{c_\eta}(\eta_c - \eta)^{1/2}\frac{GM}{R^3}. \tag{76}$$

For $\alpha > \alpha_c$, the eigenvalue $\lambda = \pm\sqrt{\lambda^2}$ can be positive implying instability. For $\alpha < \alpha_c$, the eigenvalue $\lambda = \pm i\sqrt{-\lambda^2}$ is purely imaginary so that the time evolution of the perturbation has an oscillatory nature. Close to the critical point, the pulsation period is given by

$$\omega = \frac{1}{\sqrt{c_\eta}}(\eta_c - \eta)^{1/4}\left(\frac{GM}{R^3}\right)^{1/2}. \tag{77}$$

7 Conclusion

In this paper, we have obtained the asymptotic expressions of the blow-up time and relaxation time of self-gravitating Brownian particles and biological populations close to the critical point of collapse in $d = 3$. An excellent agreement is obtained by solving numerically the Smoluchowski-Poisson system. This study confirms and improves the results obtained in [7] on the basis of heuristic arguments. A possible extension of this work would be to consider the case of polytropic distributions arising in case of anomalous diffusion [10]. More generally, we could consider the generalized Smoluchowski equation (81) proposed in [17] which is valid for an arbitrary equation of state. This equation can provide (among other applications) a generalized chemotactic model valid when the diffusion coefficient or the drift term depends on the density. These extensions will be considered in future works.

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